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# Centralizers and Monoids in Mathematical Clone Theory (New Aspects of Theoretical Computer Science)

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# Centralizers and Monoids in Mathematical Clone Theory

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## Abstract

A clone is a set of multi-variable operations which contains all projections and is closed under composition. For a set  $S$  of operations the centralizer  $S^*$  is the clone of operations which commute with all operations in  $S$ .

In this paper we consider some monoids consisting of unary operations and the centralizers of such monoids. We study (1) the centralizers of some monoids which contain the symmetric group, (2) the centralizers of permutation groups and (3) some relatively small monoids whose centralizer is the least clone.

**Keywords:** Clone; centralizer; monoid

## 1 Preliminaries

Let  $k = \{0, 1, \dots, k-1\}$  for a fixed  $k > 1$ . For a positive integer  $n$  let  $\mathcal{O}_k^{(n)}$  be the set of all  $n$ -ary operations on  $k$ , that is, maps from  $k^n$  into  $k$ , and let  $\mathcal{O}_k = \bigcup_{n=1}^{\infty} \mathcal{O}_k^{(n)}$ . Denote by  $\mathcal{J}_k$  the set of all **projections**  $e_i^n$ ,  $1 \leq i \leq n$ , over  $k$  where  $e_i^n$  is defined as  $e_i^n(x_1, \dots, x_i, \dots, x_n) = x_i$  for every  $(x_1, \dots, x_n) \in k^n$ .

**Definition 1.1** A subset  $C$  of  $\mathcal{O}_k$  is a **clone** on  $k$  if the following conditions are satisfied:

- (i)  $C$  contains  $\mathcal{J}_k$ .
- (ii)  $C$  is closed under (functional) composition.

The set of all clones on  $k$  is denote by  $\mathcal{L}_k$ . The set  $\mathcal{L}_k$  ordered by inclusion is called the **lattice of clones** on  $k$  and is denoted by  $\mathcal{L}_k$ .

The structure of  $\mathcal{L}_2$  is completely known ([Po 41]); however, the structure of  $\mathcal{L}_k$  for every  $k \geq 3$  is extremely complex and still mostly unknown. The cardinality of the lattice of clones is

known for each  $k \geq 2$ :  $|\mathcal{L}_2| = \aleph_0$  ([Po 41]) and  $|\mathcal{L}_k| = 2^{\aleph_0}$  if  $3 \leq k < \aleph_0$  ([IM 59]).

To get better understanding of the lattice  $\mathcal{L}_k$ , we started in [MMR 01] to study the clones corresponding to monoids of unary operations on  $k$  by a natural Galois connection.

This paper is an attempt to briefly summarize the work of [MMR 01], [MMR 02] and [MR 03].

**Definition 1.2** A subset  $M$  of  $\mathcal{O}_k^{(1)}$  is a **transformation monoid** (or, simply, a **monoid**) on  $k$  if it satisfies the following conditions:

- (i) For any  $s, t \in M$ , the composition  $t \circ s$  of  $s$  and  $t$  belongs to  $M$ .
- (ii) The identity operation  $\text{id}_k (= e_1^1)$  belongs to  $M$ .

Denote by  $\mathcal{M}_k$  the set of all monoids on  $k$  ordered by inclusion.

We say that operations  $f \in \mathcal{O}_k^{(m)}$  and  $g \in \mathcal{O}_k^{(n)}$  **commute** (or **permute**), in symbol  $f \perp g$ , if for every  $m \times n$  matrix  $B = (b_{ij})$  over  $k$

$$\begin{aligned} & f(g(b_{11}, \dots, b_{1n}), \dots, g(b_{m1}, \dots, b_{mn})) \\ &= g(f(b_{11}, \dots, b_{m1}), \dots, f(b_{1n}, \dots, b_{mn})). \end{aligned}$$

Clearly, the relation  $\perp$  on  $\mathcal{O}_k$  is symmetric.

**Definition 1.3** For a subset  $F$  of  $\mathcal{O}_k$ , define

$$F^* = \{ g \in \mathcal{O}_k \mid g \perp f \text{ for all } f \in F \}.$$

The set  $F^*$  is called the **centralizer** of  $F$ .

**Note:** (1) It is easy to see that  $F^*$  is a clone on  $k$  for any subset  $F$  of  $\mathcal{O}_k$ .

(2) Clearly from the definition,  $F \subseteq G$  implies  $G^* \subseteq F^*$  for any subsets  $F, G$  of  $\mathcal{O}_k$ .

**Definition 1.4** For a positive integer  $h$ , an  $h$ -ary relation on  $\mathbf{k}$  is a subset  $\rho$  of  $\mathbf{k}^h$ , i.e., a set of  $h$ -tuples over  $\mathbf{k}$ . An operation  $f \in \mathcal{O}_k^{(m)}$  preserves  $\rho$  if, for every  $h \times m$  matrix  $B = (b_{ij})$  over  $\mathbf{k}$  whose column vectors all belong to  $\rho$ , the  $h$ -tuple of values of  $f$  on the rows of  $B$  also belongs to  $\rho$ ; i.e.,

$$(f(b_{11}, \dots, b_{1m}), \dots, f(b_{h1}, \dots, b_{hm})) \in \rho$$

whenever  $(b_{1j}, \dots, b_{hj}) \in \rho$  for all  $j = 1, \dots, m$ .

For an  $h$ -ary relation  $\rho$  on  $\mathbf{k}$  set

$$\text{Pol } \rho = \{f \in \mathcal{O}_k \mid f \text{ preserves } \rho\}.$$

It is easy to see that  $\text{Pol } \rho$  is a clone on  $\mathbf{k}$ .

In this paper we shall mostly deal with the following special type of relations. For  $f \in \mathcal{O}_k^{(m)}$  set

$$f^\square = \{(a_1, \dots, a_m, f(a_1, \dots, a_m)) \mid a_1, \dots, a_m \in \mathbf{k}\}.$$

Even more specifically, our main concern is the case  $m = 1$ , that is, the case of binary relations induced by unary operations. To repeat, for an operation  $s \in \mathcal{O}_k^{(1)}$  we set

$$s^\square = \{(x, s(x)) \mid x \in \mathbf{k}\}.$$

For  $f \in \mathcal{O}_k^{(n)}$  and  $s \in \mathcal{O}_k^{(1)}$ ,  $f \in \text{Pol}(s^\square)$  is equivalent to

$$f(s(x_1), s(x_2), \dots, s(x_n)) = s(f(x_1, x_2, \dots, x_n))$$

for every  $(x_1, x_2, \dots, x_n) \in \mathbf{k}^n$ . To put it in algebraic terminology,  $f \in \text{Pol}(s^\square)$  means that  $s$  is an endomorphism of the algebra  $\langle \mathbf{k}; f \rangle$ .

From the definition of  $f \perp g$  and the symmetry of  $\perp$  we obtain for  $f, g \in \mathcal{O}_k$

$$f \perp g \iff f \in \text{Pol } g^\square \iff g \in \text{Pol } f^\square$$

and hence the centralizer  $F^*$  of  $F \subseteq \mathcal{O}_k$  is

$$F^* = \bigcap_{f \in F} \text{Pol } f^\square.$$

For  $F \subseteq \mathcal{O}_k$ ,  $\langle F \rangle$  denotes the clone generated by  $F$ , that is,  $\langle F \rangle$  is the least clone containing  $F$ .

The following definition and theorem are from [Da77, 79].

**Definition 1.5** For  $f \in \mathcal{O}_k^{(n)}$  and  $\Sigma \subseteq \mathcal{O}_k$ , the operation  $f$  is **parametrically expressible**, in

short, **p-expressible**, by  $\Sigma$  if there exist (i) integers  $m \geq 1$  and  $\ell \geq 0$ , and (ii)  $(m + \ell + 1)$ -ary operations  $g_i, h_i$  ( $i = 1, \dots, m$ ) in  $\langle \Sigma \rangle$  such that  $f^\square$  consists of all  $(x_1, \dots, x_{n+1}) \in \mathbf{k}^{n+1}$  satisfying the system of equations

$$g_i(x_1, \dots, x_{n+\ell+1}) = h_i(x_1, \dots, x_{n+\ell+1}) \quad (i = 1, \dots, m)$$

for some  $x_{n+2}, \dots, x_{n+\ell+1} \in \mathbf{k}$ , i.e.,

$$f^\square = \{(x_1, \dots, x_n, x_{n+1}) \mid \exists x_{n+2}, \dots, x_{n+\ell+1} \in \mathbf{k} \quad \forall i \in \{1, \dots, m\} \quad g_i(x_1, \dots, x_{n+\ell+1}) = h_i(x_1, \dots, x_{n+\ell+1})\}.$$

**Theorem 1.1** (Kuznetsov criterion) Let  $f \in \mathcal{O}_k$  and  $\Sigma \subseteq \mathcal{O}_k$ . Then  $f$  is p-expressible by  $\Sigma$  if and only if  $\Sigma^* \subseteq \{f\}^*$ .

This property turns out to be quite useful in the study of clones, in particular, of centralizers.

## 2 Centralizers of Monoids containing the Symmetric Group $S_k$

### 2.1 The Sequence $\{N_i\}$ of Monoids

First we consider the centralizers of monoids which sit “above” the symmetric group  $S_k$ , i.e., monoids containing  $S_k$ .

In [MMR 01] we introduced the operations  $d_i$ 's as follows: For every  $i = 0, 1, \dots, k-1$ , a unary operation  $d_i : \mathbf{k} \rightarrow \mathbf{k}$  is defined by

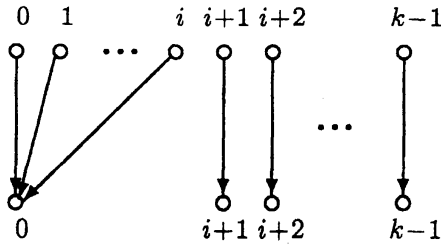
$$d_i(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq i, \\ x & \text{if } i < x \leq k-1. \end{cases}$$

In particular,  $d_0 = e_1^1$  (the identity operation on  $\mathbf{k}$ ) and  $d_{k-2}$  satisfies the following:  $d_{k-2}(x) = 0$  for  $x \in \mathbf{k} \setminus \{k-1\}$  and  $d_{k-2}(k-1) = k-1$ .

The monoid  $N_i$  for  $i \in \mathbf{k}$  is defined as the clone generated by the symmetric group  $S_k$  on  $\mathbf{k}$  and the operation  $d_i$ , i.e.,

$$N_i = \langle S_k \cup \{d_i\} \rangle.$$

It is easy to see that  $N_0 = S_k$  and  $N_1 = \mathcal{O}_k^{(1)}$ . Also,  $N_{k-1}$  consists of  $S_k$  and all constant unary operations.

Figure 1. The unary operation  $d_i$ 

We have a sequence

$$S_k \subset N_{k-1} \subset N_{k-2} \subset \dots \subset N_3 \subset N_2 \subset \mathcal{O}_k^{(1)}$$

where the symbol  $\subset$  denotes proper inclusion.

We also prepare the following unary operations.

Let  $\sigma, \tau : k \rightarrow k$  be the permutations defined as

$$\sigma(x) = \begin{cases} x+1 & \text{if } x \in k \setminus \{k-1\} \\ 0 & \text{if } x = k-1 \end{cases}$$

and

$$\tau(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \\ x & \text{otherwise,} \end{cases}$$

i.e., in the cyclic notation  $\sigma = (0 \ 1 \ \dots \ k-1)$  and  $\tau = (0 \ 1)$ .

Let  $c_0 : k \rightarrow k$  be a constant operation defined as  $c_0(x) = 0$  for all  $x \in k$ .

For every  $i \in k$  let  $\chi_i : k \rightarrow k$  be the unary operation defined as

$$\chi_i(x) = \begin{cases} 1 & \text{if } x = i, \\ 0 & \text{otherwise.} \end{cases}$$

Set  $\Gamma_k = \{\chi_i \mid i \in k\}$ .

**Lemma 2.1** For every  $i \in k$ ,  $\chi_i$  is generated by  $\{\sigma, d_{k-2}\}$ .

**Proof** It is easily verified that

$$\chi_i = \sigma d_{k-2} \sigma^{-1} d_{k-2} \sigma^{-(i+1)}. \quad \square$$

## 2.2 Application of the Kuznetsov Criterion : $N_i^* = N_{k-2}^*$

**Proposition 2.2** For  $1 \leq i \leq k-2$ , the operation  $d_i$  is  $p$ -expressible from  $N_{k-2}$ , namely,

$$d_i^\square = \{(x, y) \in k^2 \mid c_0(x) = \chi_1(y) = \dots = \chi_i(y), \\ \chi_{i+1}(x) = \chi_{i+1}(y), \dots, \chi_{k-1}(x) = \chi_{k-1}(y)\}.$$

**Proof** Denote by  $\rho_i$  the binary relation on the right side of the above equation.

For each  $x = 0, 1, \dots, k-1$ , we consider which  $y$  satisfies  $(x, y) \in \rho_i$  where  $y \in \{0, \dots, k-1\}$ .

Case 0: Suppose  $x = 0$ . Then it is easy to see that  $c_0(0) = \chi_1(y)$  implies  $y \neq 1$ . Similarly, it is implied from  $c_0(0) = \chi_j(y)$  that  $y \neq j$  for every  $j = 1, \dots, i$ . Next, from  $\chi_{i+1}(0) = \chi_{i+1}(y)$  it is implied that  $y \neq i+1$ . Similarly, it is implied from  $\chi_\ell(0) = \chi_\ell(y)$  that  $y \neq \ell$  for every  $\ell = i+1, \dots, k-1$ . So the only possibility is  $y = 0$ , and clearly  $(0, 0) \in \rho_i$ .

Analogously, we can check, for every  $x = 0, \dots, i$ , that  $(x, 0)$  is the only pair that satisfies  $(x, y) \in \rho_i$  for  $y = 0, \dots, k-1$ .

Case  $i+1$ : Suppose  $x = i+1$ . In this case  $c_0(i+1) = \chi_j(y)$  implies that  $y \neq j$  for every  $j = 1, \dots, i$ . Now,  $\chi_{i+1}(i+1) = \chi_{i+1}(y)$  implies  $y = i+1$ . And,  $\chi_\ell(i+1) = \chi_\ell(y)$  implies  $y \neq \ell$  for every  $\ell = i+2, \dots, k-1$ . Hence  $(i+1, i+1)$  is the only pair that belongs to  $\rho_i$  with  $i+1$  as the first component.

In the similar way we can check, for every  $x = i+1, \dots, k-1$ , that  $(x, x)$  is the only pair that satisfies  $(x, y) \in \rho_i$  for  $y = 0, \dots, k-1$ .

Thus we have proved that  $\rho_i = d_i^\square$ .  $\square$

**Corollary 2.3** For every  $i$ ,  $1 \leq i \leq k-2$ ,

$$N_i^* = N_{k-2}^*.$$

**Proof** Clear from the Kuznetsov criterion, Lemma 2.1 and Proposition 2.2.  $\square$

## 2.3 The Centralizers of $N_i$ 's

In this subsection we assert without proof that  $(\mathcal{O}_k^{(1)})^* = \mathcal{J}_k$ . Then, as a corollary, we have  $N_i^* = \mathcal{J}_k$  for every  $1 \leq i \leq k-2$ .

An operation  $f \in \mathcal{O}_k^{(n)}$  is called **conservative** if  $f(a_1, \dots, a_n) \in \{a_1, \dots, a_n\}$  for every  $(a_1, \dots, a_n) \in k^n$ .

**Lemma 2.4** For  $f \in \mathcal{O}_k^{(n)}$ , if  $f \in (\mathcal{O}_k^{(1)})^*$  then  $f$  is conservative.

Moreover, we can show the following:

**Proposition 2.5** For  $f \in \mathcal{O}_k^{(n)}$ , if  $f \in (\mathcal{O}_k^{(1)})^*$  then  $f$  is a projection. I.e.,  $(\mathcal{O}_k^{(1)})^* = \mathcal{J}_k$ .

The proof is tedious and omitted.

**Corollary 2.6** For every  $i = 1, 2, \dots, k-2$ ,

$$N_i^* = \mathcal{J}_k.$$

To summarize, we have the following:

$$N_0^* (= S_k^*) = \{f \in \mathcal{O}_k \mid f \text{ is synchronous}\}$$

$$N_{k-1}^* = \{f \in \mathcal{O}_k \mid f \text{ is synchronous and idempotent}\}$$

$$N_{k-2}^* = \dots = N_2^* = N_1^* (= (\mathcal{O}_k^{(1)})^*) = \mathcal{J}_k$$

(The definition of a “synchronous” operation appears in the next section.)

### 3 The Centralizers of the Subgroups of $S_k$

Next we turn to the centralizers of monoids which sit “below” the symmetric group  $S_k$ , i.e., monoids contained in  $S_k$ , that is, permutation groups.

#### 3.1 The Centralizers of the Permutation Groups

As we have seen in the previous section, many different monoids have the same clone  $\mathcal{J}_k$  as their centralizer. But, surprisingly, the situation is quite different for permutation groups on  $k$ , i.e., subgroups of the symmetric group  $S_k$ . In fact, the  $*$ -operator is proved to be an injective mapping on the set of permutation groups.

**Lemma 3.1** For any subgroup  $H$  of  $S_k$  and any  $s \in S_k$ , if  $s^\square$  is  $p$ -expressible by  $H$  then  $s \in H$ .

**Proof** If  $s^\square$  is  $p$ -expressible by  $H$ , which is a subgroup of  $S_k$ , then  $s^\square$  can be expressed as

$$s^\square = \{(x, y) \in k^2 \mid t(x) = u(y)\}$$

for some  $t, u \in H$ . This equation can be rephrased as

$$s^\square = \{(x, y) \in k^2 \mid (u^{-1}t)(x) = y\}.$$

This implies  $s = u^{-1}t$  and thus  $s \in H$  as desired.  $\square$

**Theorem 3.2** For any subgroups  $H_1$  and  $H_2$  of  $S_k$ , if  $H_1 \neq H_2$  then  $H_1^* \neq H_2^*$ .

**Proof** Suppose that  $H_1 \neq H_2$  and  $H_1^* = H_2^*$ . Without loss of generality we may assume that there exists a unary operation  $s \in H_2 \setminus H_1$ . Then it holds that

$$H_1^* (= H_2^*) \subseteq (s^\square)^* (= \text{Pol } s^\square),$$

since

$$H_2^* = \bigcap_{t \in H_2} \text{Pol } t^\square.$$

It follows from the Kuznetsov criterion that  $s^\square$  is  $p$ -expressible by  $H_1$ . Hence by Lemma 3.1 we have  $s \in H_1$ , a contradiction.  $\square$

#### 3.2 The Centralizer of the Symmetric Group $S_k$

Marczewski [Ma 64] initiated the study of homogeneous algebras and homogeneous operations. An operation  $f \in \mathcal{O}_k$  is *homogeneous* if it belongs to the centralizer of the symmetric group  $S_k$ .

**Definition 3.1** For  $(x_1, \dots, x_n), (y_1, \dots, y_n) \in k^n$ , we say that  $(x_1, \dots, x_n)$  is *similar* to  $(y_1, \dots, y_n)$  if the following condition is satisfied:

$$x_i = x_j \iff y_i = y_j \quad \text{for any } 0 \leq i, j < n.$$

**Definition 3.2** An operation  $f \in \mathcal{O}_k^{(n)}$  is *synchronous* if the following condition is satisfied for any element  $(x_1, \dots, x_n)$  in  $k^n$ .

If  $|\{x_1, \dots, x_n\}| \neq k-1$  then

- (1)  $f(x_1, \dots, x_n) = x_\ell$  for some  $1 \leq \ell \leq n$ , and
- (2)  $f(y_1, \dots, y_n) = y_\ell$  for any  $(y_1, \dots, y_n) \in k^n$  which is similar to  $(x_1, \dots, x_n)$ ,

and, if  $|\{x_1, \dots, x_n\}| = k-1$  and  $f(x_1, \dots, x_n) = u$  for some  $u \in k$  then

- (1)  $u = x_\ell$  for some  $1 \leq \ell \leq n$  implies  $f(y_1, \dots, y_n) = y_\ell$  for any  $(y_1, \dots, y_n) \in k^n$  which is similar to  $(x_1, \dots, x_n)$ , and
- (2)  $u \in k \setminus \{x_1, \dots, x_n\}$  implies  $f(y_1, \dots, y_n) = v$  where  $v \in k \setminus \{y_1, \dots, y_n\}$  for any  $(y_1, \dots, y_n) \in k^n$  which is similar to  $(x_1, \dots, x_n)$ .

The set of all synchronous operations in  $\mathcal{O}_k$  is denoted by  $\text{SYN}_k$ .

**Theorem 3.3** For the symmetric group  $S_k$  on  $k$ ,

$$S_k^* = \mathcal{SYN}_k,$$

that is, an operation  $f \in \mathcal{O}_k$  is homogeneous if and only if it is synchronous.

Again, we skip the proof. (See [MMR 01].)

### 3.3 The Centralizer of the Alternating Group $A_k$

For the alternating group  $A_k$  on  $k$ , Theorem 3.2 implies  $S_k^* \neq A_k^*$ . In the following we show how the centralizer  $A_k^*$  is characterized.

**Theorem 3.4** Let  $n \geq 1$  and  $f \in \mathcal{O}_k^{(n)}$ . Then  $f \in A_k^*$  if and only if the following conditions are satisfied for every  $h$ -block equivalence relation  $\theta$  on  $\{1, \dots, n\}$  with  $h \leq k$ :

- (1) If  $h \leq k - 3$  then  $f$  is a projection on  $\Pi_n(\theta)$ .
- (2) If  $h = k - 2$  then on  $\Pi_n(\theta)$  the operation  $f$  is either a projection or  $f$  is uniquely determined by  $f(a) \in \{k - 2, k - 1\}$  where  $a \in \Pi_n(\theta)$  satisfies  $\text{sgn } a = (0, 1, \dots, k - 3)$ .
- (3) If  $h = k - 1$  then for  $i = 0, 1$  the operation  $f$  on  $H_{k-1,i}$  is either a projection or it is uniquely determined by  $f(a^i) = k - 1$ .
- (4) If  $h = k$  then for  $i = 0, 1$  the operation  $f$  on  $H_{k,i}$  is a projection.  $\square$

For the proof refer to [MR 03].

## 4 Monoid whose Centralizer is $\mathcal{O}_k$

Here we look at the largest clone in the lattice  $\mathcal{L}_k$ , that is, the clone  $\mathcal{O}_k$ . It is obvious that the least monoid  $\{\text{id}\}$  consisting only of the identity operation corresponds to  $\mathcal{O}_k$ . We show that no other monoid corresponds to  $\mathcal{O}_k$ .

**Proposition 4.1** For any monoid  $M \subseteq \mathcal{O}_k^{(1)}$ ,  
 $M^* = \mathcal{O}_k$  if and only if  $M = \{\text{id}\}$ .

**Proof** ( $\Leftarrow$ ) Clear. ( $\Rightarrow$ ) Let  $M^* = \mathcal{O}_k$ . Then by Proposition 2.5 we have

$$\text{id} \in M \subseteq M^{**} = \mathcal{O}_k^* \subseteq (\mathcal{O}_k^{(1)})^* = \mathcal{J}_k$$

which shows  $M = \{\text{id}\}$ .  $\square$

## 5 “Small” Monoids whose Centralizer is $\mathcal{J}_k$

Since  $*$ -operator induces a Galois connection between monoids and clones, it seems natural to consider that a smaller monoid corresponds to a larger clone and vice versa. However, relatively small monoids were found whose centralizer is  $\mathcal{J}_k$ , the smallest clone.

### 5.1 Application of the Kuznetsov Criterion : $\langle \{\sigma, d_{k-2}\} \rangle^* = N_{k-2}^*$

Now we look for small monoids whose centralizer is  $\mathcal{J}_k$ .

**Proposition 5.1** The transposition  $\tau$  is  $p$ -expressible from  $\{\sigma, d_{k-2}\}$ , namely,

$$\tau^\square = \{(x, y) \in k^2 \mid \chi_0(x) = \chi_1(y), \chi_1(x) = \chi_0(y), \\ \chi_2(x) = \chi_2(y), \dots, \chi_{k-1}(x) = \chi_{k-1}(y)\}.$$

**Proof** Similar to the proof of Proposition 2.2.  $\square$

**Corollary 5.2** Let  $M_1 = \langle \{\sigma, d_{k-2}\} \rangle$ . Then

$$M_1^* = N_{k-2}^* (= \mathcal{J}_k).$$

**Proof** It is well-known that  $\{\sigma, \tau\}$  generates the symmetric group  $S_k$ . So the assertion is clear from Proposition 5.1.  $\square$

**Remark:** (1) The permutations in  $M_1$  are those generated by  $\sigma$ , that is,  $M_1 \cap S_k = \langle \sigma \rangle$ . Thus, not all permutations are contained in  $M_1$ , i.e.,  $M_1 \not\supseteq S_k$ .

(2) An implication from above is that a monoid  $M$  does not need to contain all permutations in order to satisfy  $M^* = \mathcal{J}_k$ .

### 5.2 Application of the Kuznetsov Criterion : $\Gamma_k^* = N_{k-2}^*$

It should be reminded that  $\Gamma_k = \{\chi_i \mid i \in k\}$ .

Analogous constructions to the one in the preceding subsection give us the following:

**Proposition 5.3** The transposition  $(ij)$  ( $i \neq j$ ) is  $p$ -expressible from  $\Gamma_k$ .

**Proof** A construction for  $\tau = (01)$  was given in Proposition 5.1. Any transposition  $(ij)$  can be constructed similarly.  $\square$

## Corollary 5.4

$$\Gamma_k^* = N_{k-2}^* (= \mathcal{J}_k).$$

**Proof** It is well-known that the set of all transpositions generates the symmetric group  $S_k$ . So the assertion is clear.  $\square$

**Remark:** (1)  $\Gamma_k$  consists only of two-valued operations. In particular,  $\Gamma_k$  has no permutations, i.e.,  $\Gamma_k \cap S_k = \emptyset$ .

(2) An implication from above is that transitivity is not necessarily required for a monoid  $M$  to satisfy  $M^* = \mathcal{J}_k$ .

5.3 The Centralizer of  $\langle \Gamma_k \rangle \setminus \{\chi_0, \bar{\chi}_1\}$ 

Even a smaller monoid than  $\Gamma_k$  can be proved to have  $\mathcal{J}_k$  as its centralizer.

The monoid  $\langle \Gamma_k \rangle$  generated by  $\Gamma_k$  consists of the following operations:

$$\langle \Gamma_k \rangle = \{\chi_0, \chi_1, \dots, \chi_{k-1}, \bar{\chi}_0, \bar{\chi}_1, \dots, \bar{\chi}_{k-1}, c_0, c_1, \text{id}_k\}$$

where  $\bar{\chi}_i(x) = 1 - \chi_i(x)$  and  $c_i(x) = i$  for all  $i = 0, 1, \dots, k-1$ .

Set  $\Theta_k = \langle \Gamma_k \rangle \setminus \{\chi_0, \bar{\chi}_1\}$ , that is,

$$\Theta_k = \{\chi_1, \dots, \chi_{k-1}, \bar{\chi}_0, \bar{\chi}_2, \dots, \bar{\chi}_{k-1}, c_0, c_1, \text{id}_k\}.$$

It is readily verified that  $\Theta_k$  is a submonoid of  $\langle \Gamma_k \rangle$ .

For this monoid  $\Theta_k$  we can prove the following:

**Proposition 5.5** *Let  $k > 2$ . Then  $\Theta_k^* = \mathcal{J}_k$ .*

Thus the centralizer  $\Theta_k^*$  of  $\Theta_k$  is the least clone  $\mathcal{J}_k$ . The proof is long and omitted here. (See [MR 03])

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